BEAMSPACE IQML

Zhi Tian, Harry L. Van Trees

Center of Excellence in C3I
George Mason University
Fairfax, VA 22030-4444, USA
ztian@gmu.edu, hlv@gmu.edu

ABSTRACT

Iterative quadratic maximum likelihood (IQML) is an efficient approach for estimating the directions of arrival (DOA) of signals impinging on a uniform linear array. It is desirable to apply the IQML method in beamspace so as to take advantage of the benefits of beamspace operations in terms of reduced computational complexity and improved resolution thresholds. The difficulty with beamspace IQML is that the Vandermonde structure of the array manifold may not preserve during the beamspace transformation. We tackle this obstacle using discrete Fourier transform (DFT) beams and derive a parameterized polynomial in the beamspace dimension. The proposed beamspace IQML algorithm is computationally efficient, and provides satisfactory estimation results.

Key words: IQML estimation, beamspace processing, polynomial parameterizations, discrete Fourier transform matrices

1. INTRODUCTION

Iterative quadratic maximum likelihood (IQML) [1] is a well-known method that deals with direction estimation of signals impinging on a uniform linear sensor array. It offers an efficient implementation to the deterministic maximum likelihood estimator and also shares the same properties in low noise cases. The development of the IQML algorithm hinges on the polynomial parameterization of the projection matrix that is orthogonal to the signal subspace, yielding a computationally attractive method for computing the likelihood function as expressed by a linear representation of the orthogonal subspace.

Operation in beamspace offers potential performance improvement over operation in element space, including reduced computational complexity, better resolution threshold, and reduced sensitivity to sensor perturbations. It is desirable to develop beamspace IQML so that high-resolution estimation with low computational load can be achieved. The difficulty with beamspace IQML is that the Vandermonde structure of the array manifold may not preserve during the beamspace transformation. It is therefore not straightforward to parameterize the orthogonal subspace by a linear polynomial that describes the spatial characteristics of the signal components. Lee and Wengrovitz [2] studied beamforming matrices that were constructed from banded Toeplitz matrices, and proposed a procedure for constructing a reduced directional spectrum of minimum polynomial degree for subspace-based techniques such as MUSIC and min-Norm. Zolotowski [3] presented an IQML algorithm in the beamspace domain where the beams were formed from the outputs of identical, adjacent, overlapping subarrays. Zolotowski [3] [4] also developed 2-D and 3-D beamspace domain maximum likelihood bearing estimation schemes for low-angle radar tracking, in which the number of mutually orthogonal beams was specified to be exactly one more than the number of sources.

In this paper, we develop a beamspace IQML estimator using the most commonly used discrete Fourier transform (DFT) matrix beamformers. The properties of DFT matrices are discussed in details in [5] where a beamspace root-MUSIC algorithm is derived. We exploit the common out-of-band nulls property exhibited by DFT beams to extend the element-space IQML result to the beamspace. Proper matrix transformation is designed to preserve the Vandermonde structure of array manifold in the beamspace. Different from [4], we do not regulate the relationship between the number of beams and the number of sources. DFT matrices do not belong to the beamformer classes that are studied in [2] or [3], although they are related to one another via unitary transformation.

This paper is organized as follows. In the next section the data model is introduced and the difficulty in developing a beamspace IQML algorithm is pointed out. We derive a linear representation of the noise space in beamspace in section 3 and provide the beamspace IQML formulation in section 4. Computation of relevant parameters is discussed in section 5. Performance comparison with the elementspace IQML algorithm and the most acclaimed beamspace ESPRIT algorithm are presented by computer simulations in section 6, and a summary is given in section 7.
2. PROBLEM STATEMENT

Let $x_k$ denote the $N \times 1$ vector of signals received by a uniform linear array of $N$ sensors at the $k$-th snapshot. The vector $x_k$ is given by

$$x_k = V_N(u_k) s_k + n_k,$$

where $s_k \in \mathbb{C}^{D \times 1}$ contains $D$ narrowband transmitted signals and the zero mean Gaussian noise process $n_k \in \mathbb{C}^{N \times 1}$ is spatially and temporally white. $V_N(u)$ is the $N \times D$ array response matrix whose $i$-th column is given by

$$v_N(u_i) = [1 e^{j2\pi \Delta u_1} \cdots e^{j(N-1)2\pi \Delta u_1}]^T,$$

where $u_i$ is the $i$-th direction of arrival in $u$-space, and $\Delta$ is the element spacing measured in wavelengths.

Consider a beamspace operation denoted by its beamforming matrix $W$. We require $W^H W = I$ so that the noise remains white in the beamspace output. This is always achievable by whitening of non-orthogonal beams. The output at the beamspace matrix is given by

$$y_k = W^H x_k = V_{bs}(u) s_k + n_{bs,k}$$

where $V_{bs}(u) = W^H V_N(u)$ is made up of beamspace response vectors $v_{bs}(u_i) = W^H v_N(u_i)$ for $i = 1, \cdots, D$. The beamspace output $y_k$ is assumed to be white in time with spatial covariance

$$R_{bs} = E\{y_k y_k^H\} = V_{bs}(u) S V_{bs}(u)^H + \sigma_n^2 I,$$

where $S = E\{s_k s_k^H\}$ denotes the covariance of the transmitted signal $s_k$. An estimate of $R_{bs}$ is given by $\hat{R}_{bs} = K^{-1} \sum_{k=1}^K y_k y_k^H$, where $K$ is the number of array snapshots available.

In the IQML algorithm, the spatial characteristics of the signal components are parameterized by a coefficient vector $b = [b_0 \cdots b_D]^T$. $b$ is defined such that the polynomial

$$b(z) = b_D z^D + b_{D-1} z^{D-1} + \cdots + b_0$$

has $D$ roots at $z_i = e^{j2\pi u_i}$, $i = 1, \cdots, D$. The ML estimate of $u_i$ can be obtained from the ML estimate of $b$. In order to extend the element-space IQML algorithm to the beamspace domain, it is necessary to find a linear parameterization of the null space of $V_{bs}$. This is equivalent to finding a full rank matrix $B(b)$ such that

$$\mathcal{L}(B) V_{bs} = 0.$$  \hspace{1cm} (6)

The operator $\mathcal{L}\{\cdot\}$ represents a linear transform operation, and $B$ is a $N_{bs} \times (N_{bs} - D)$ Toeplitz matrix given by

$$B^H = \begin{bmatrix} b_D & \cdots & b_0 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & b_D & \cdots & b_0 \end{bmatrix}.$$ \hspace{1cm} (7)

Polynomial parameterization by (6) is not always possible for any beamforming transformation. In the element space, the linear parameterization of the noise space is made possible by the Vandermonde property of the element-space array response matrix. The Vandermonde structure of the array manifold may not be preserved after beamspace transformation. A crucial step in developing the beamspace IMQL algorithm is to find the linear parameterization formula $\mathcal{L}\{B\}$.

3. POLYNOMIAL PARAMETERIZATION OF DFT BEAMFORMING MATRICES

The most common beamspace matrix has rows that consist of conventional beams whose pointing directions are spaced at $2/N$ intervals in $u$-space. This is also called a discrete Fourier transform (DFT) beamforming matrix. Due to the common out-of-band nulls existing in DFT beams, a DFT beamforming matrix can be transformed into a banded Toeplitz matrix [5]. This transformation preserves the Vandermonde property of the beamspace array manifold and enables the polynomial parameterization in the beamspace.

A beamspace matrix consisting of $N_{bs}$ DFT beamforming vectors are given by

$$W = \frac{1}{\sqrt{N}} \begin{bmatrix} v_N(m \frac{2}{N}) & \cdots & v_N((m + N_{bs} - 1) \frac{2}{N}) \end{bmatrix}$$

where the integer number $m$ should be properly chosen so that the $N_{bs}$ beams cover most of the signal energy.

It has been shown [5] that $W$ can be factored as

$$W = C Q$$

where $Q$ is a $N_{bs} \times N_{bs}$ full-rank matrix and $C$ is a $N \times N_{bs}$ banded Toeplitz matrix

$$C = \begin{bmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & c_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & c_{N-N_{bs}} & c_0 \end{bmatrix} \begin{bmatrix} \vdots \\ c_0 \end{bmatrix}$$ \hspace{1cm} (10)

The following equality holds by arithmetic manipulations [2] [5]:

$$C^H v_N(u) = \alpha(u) v_{N_{bs}}(u)$$

where $v_{N_{bs}}(u)$ is defined by (2) with $N$ replaced by $N_{bs}$. Note that $\alpha(u) = e^{j2\pi u} c_{\xi_{N_{bs}}} \sum_{i=0}^{N-N_{bs}} c_i e^{j2\pi u}$ is a scalar, which does not affect the structure in $v_{N_{bs}}(u)$. The property in (11) is critical to the applicability of IQML in beamspace.
It follows from (5) that
\[ B^H v_{N_{bs}}(u_i) = z_i^{N - N_{bs} - 1} \begin{bmatrix} z_i^0 \\ \vdots \\ z_i^{N_{bs} - D} \end{bmatrix} b(z_i) = 0 \] (12)
for \( i = 1, \cdots, D \). Define a \( N_{bs} \times (N_{bs} - D) \) matrix \( F = Q^{-1} B \). From (9), (11) and (12), we have
\[ F^H v_{bs}(u_i) = B^H Q^{-H} W^H v_N(u_i) = B^H C^H v_N(u_i) = a(u_i) B^H v_{N_{bs}}(u_i) = 0 \] (13)
Since \( F \) has rank \( N_{bs} - D \), its columns span the orthogonal complement to the signal subspace, that is,
\[ P_{bs} = P_F, \] (14)
where \( P_F = F(F^H F)^{-1} F^H \). Therefore, we have
\[ \mathcal{L}(B) = F^H = B^H Q^{-H}. \] (15)

4. BEAMSPACE IQML

The deterministic ML estimate is given in the beam space by
\[ \hat{u} = \arg\min_u \text{tr}\left\{ P_{bs}(u) \tilde{R}_{bs} \right\} \]
\[ = \arg\min_b \text{tr}\left\{ P_F(b) \tilde{R}_{bs} \right\}. \] (16)
\( \tilde{R}_{bs} \) is the beamspace covariance matrix.
Defining \( R_Q = Q^{-H} \tilde{R}_{bs} Q^{-1} \), we have
\[ \text{tr}\left\{ P_F \tilde{R}_{bs} \right\} = \text{tr}\left\{ F(F^H F)^{-1} F^H \tilde{R}_{bs} \right\} \] (17)
\[ = \text{tr}\left\{ Q^{-1} B(F^H F)^{-1} B^H Q^{-H} \tilde{R}_{bs} \right\} \] (18)
\[ = \text{tr}\left\{ P(F(F^H F)^{-1} B^H Q^{-H}) \right\}. \] (19)
Minimization of the objective function in (19) can be readily solved by the IQML method proposed by Bresler and Macovski [1]. The detailed iterative procedure follows the standard element-space IQML algorithm and can be referred to [6].

5. FACTORIZATION OF THE DFT MATRIX

A DFT beamforming matrix may be decomposed into the product of a banded Toeplitz matrix \( C \) and a full-rank matrix \( Q \) (9). Only the inverse of the matrix \( Q \) needs to be computed for the IQML procedure. [5] provides a closed-form expression for \( Q \) without forming \( C \). Here we briefly explain an intuitive way to compute the matrix \( C \), and give an simple expression to compute \( Q^{-1} \) directly.

The \( N_{bs} \) columns of \( W \) are part of the \( N \times N \) DFT matrix. Due to the orthogonal properties of DFT beams, the other \( N - N_{bs} \) columns of the \( N \times N \) DFT matrix that are not contained in \( W \) are orthogonal to each of the \( N_{bs} \) columns of \( W \). Mathematically,
\[ W^H v_N(u_n) = 0 \] (20)
or
\[ Q^H C^H v_N(u_n) = 0 \] (21)
for \( u_n \in \{(m + N_{bs})/N, \ldots, (N + m - 1)/N\} \). Define a polynomial
\[ c(z) = c_{N - N_{bs}} z^{N - N_{bs}} + \cdots + c_0. \] (22)
(21) implies that \( c(z) \) has \( N - N_{bs} \) roots at \( z_n = \exp(j \pi u_n) \). Therefore, the coefficients of \( c(z) \) can be found by
\[ [c_0, \ldots, c_{N - N_{bs}}]^H = \text{poly}\{e^{j \pi u_n}\} \] (23)
where the operator \( \text{poly}\{ \} \) converts roots to polynomial.
Once the matrix \( C \) is computed, the inverse of the matrix \( Q \) can be found by pre-multiplying both sides of (9) by \( W^H \):
\[ W^H W = W^H C Q \]
(24)
Therefore, the inverse of \( Q \) is given by \( Q^{-1} = W^H C \).

6. PERFORMANCE EVALUATION

Computer simulations are performed using a 20-element uniform linear array with elements spaced half a wavelength apart. The impinging signals consist of two equi-powered sources which are 0.5HPBW apart from each other. The beamspace dimension is chosen to be \( N_{bs} = 8 \). We examine the performance of the proposed beamspace IQML algorithm with DFT beams as compared to the element-space IQML and the beamspace ESPRIT methods, using 100 snapshots and 500 trials. In both the element-space and the beamspace IQML algorithms, we use a linear constraint on the vector \( b \) to ensure non-trivial results.

Both the cases of uncorrelated and correlated signals are considered. For performance comparison, the root mean square estimation error (RMSE) and the probability of resolution are plotted versus the signal-to-noise ratio (SNR). The results for the uncorrelated case and the correlated case are depicted in Figure 1 and 2, respectively. The correlation magnitude in the second case is \( \rho = 0.95 \) and the phase correlation is 0.

Cramer-Rao bounds (CRB) in both the element-space and the beamspace are plotted for references. Element-space IQML has lower estimation error, comparable SNR resolution thresholds, but much higher complexity. Beamspace ESPRIT is computationally efficient but has a small offset to the Cramer-Rao bound asymptotically. The threshold behavior of the beamspace ESPRIT algorithm is more sensitive to the signal correlation than the ML algorithms. The beamspace IQML algorithm with DFT beams has the best overall performance. It not only has low computational complexity but also provides good estimation performance for both uncorrelated and correlated signals.
7. SUMMARY

This paper presents a beamspace IQML algorithm that uses the common out-of-band nulls property exhibited by DFT beams to construct a linear representation of the deterministic maximum likelihood function. This beamspace estimator not only saves computation over element-space ML methods, but also compares favorably over other suboptimal high-resolution estimators.

8. REFERENCES


Figure 1: Uncorrelated signals

Figure 2: Correlated signals, $\rho = 0.95$. 

364